Latent liquidity models: An universal mechanism for price impact

[see arXiv:1412.0141]
Outline

- **Motivation**: empirical results and phenomenological formula for anomalous price impact
- **Idea**: an intuition about latent liquidity models
- **Model**: a consistent model for anomalous price impact
- **Results**: analysis, numerics and predictions
- **Discussion**: extensions and open problems
Motivation
The problem of market impact

**A trivial statement**
Buy trades push prices up. Sell trades push prices down.

**A not-so-trivial question**
How much do buy (sell) trades move prices up (down)?
For how long?

**A hard question**
How to measure this? How to model it?
In the following, we will study:

\[ \mathcal{I}(Q) = \langle (p_T - p_0) | Q \rangle \]
Relevance of market impact

Why is this issue relevant?

- **Theory:** Relevant, because market impact is the mechanism through which prices absorb information encoded in trades, making prices efficient;

- **Practice (I):** Market impact is a cost for traders, which they need to accurately control in order to optimize execution;

- **Practice (II):** For regulators, market impact controls stability.
Evidence dating back to 1997 (!) shows that impact has a concave shape (roughly) independent of:

- Venue
- Maturity
- Historical period
- Geographical area

[see Torre (1997), Almgren et al. (2003), Moro et al. (2009), Tóth et al. (2011), Gomes, Waelbroeck (2014), Bershova, Rakhlin (2013), IM et al. (2014), X. Brokmann et al. (2014), Zarinelli et al. (2015)]

[from Tóth et al (2011), impact of ≈ 500 000 trades]
Phenomenological formula for MI

The shape of the impact function is well-fitted by the following parameterization:

\[ I(Q) = Y \sigma \left( \frac{Q}{V} \right)^{1/2} \]

- \( I(Q) \): price change
- \( Q \): executed volume
- \( \sigma \): daily volatility
- \( Y \): Y-ratio (adimensional \( \sim 1 \))
- \( V \): daily traded volume

where we notice:

- **Market fragility**: A concave dependence implies large response to small trades (1% of daily volume moves by 10% of daily volatility)
- **Non-additivity**: When executing \( 2Q \) trades sequentially, the effect of the second group of trades is smaller
- **Scale-independence**: Volatility scales with \( T^{1/2} \) and volume with \( T \). So, the law has no scale.
Idea
The origin of (mechanical) impact

Why does transient impact arise?

**Limited liquidity.** If liquidity was infinite, everyone could instantly execute, and traders wouldn’t need to split their trades and execute incrementally…

Why is liquidity limited?

**Because price exist!** Due to market clearing, you don’t have *any* liquidity at the current price by definition.

We want to incorporate these *generic (universal)* ingredients in a stylized model, in order to see how much of the phenomenon we can capture.
Particle price vs interface price

Existing orders are arranged on a price line: their interface defines the price

The actual microstructure motivates a model in which prices are not fundamental, but emergent objects: prices evolve through orders.
Intuition for concave impact

Let’s zoom in around the current price. For a non-degenerate density,

\[ \rho_A(x - p_t) \approx \rho_B(p_t - x) \approx \mathcal{L}x \]

QUESTION:
How much does the instant execution of a volume Q affect the price?

\[ Q = \pm \int_{p_t}^{p_t \pm \Delta p} \rho_{A/B}(x) = \mathcal{L} \frac{\Delta p^2}{2} \]

\[ \Delta p = \sqrt{\frac{2Q}{\mathcal{L}}} \]
Orders of magnitude estimation...

What we stated in the last slide is valid in a specific setting:

\[ \nu^{-1} \gg T \]

renewal time of the demand/supply curve

duration of the order

otherwise you expect the impact to cross-over to a trivial, linear region.

The correct interpretation of the demand and supply curve cannot be the order book!

\[ T \sim \text{ (hours)} \]

\[ \nu^{-1}_{OB} \sim \text{ (seconds)} \]

Need for a notion of latent liquidity

\[ \nu^{-1} \sim \text{ (hours)} \]
Latent order book

Latent Bid
Latent Ask
Bid
Ask

Evidence: \[\text{Empirical: daily traded volume vs instantly available volume}\]
\[\text{Theoretical: no incentive in giving away private information}\]

Latent vs Real

\[\text{\sim 10^2}\]

Some history…

- **ε-intelligence model**: first implementation of this idea, realistic market model yielding concave impact [B Tóth, Y Lempérière, C Deremble, J de Lataillade, J Kockelkoren, J-P Bouchaud, Phys. Rev. X (2011)]

- **Its generalizations**: analysis of generalization of the model: sqrt impact result is robust. Many issues addressed (impact of limit orders, adaptive order flux) [IM, B Tóth, J-P Bouchaud, Phys. Rev. E. 89 (2014)]

- **Stylized model**: extracted a stylized version of the model in which everything can be computed analytical. Impact is exactly found to be sqrt. [IM, B Tóth, J-P Bouchaud, Phys. Rev. Lett. (2014)]

- **This model**: Bridges the gap among the two approaches: still amenable to analytical calculations, but allows realistic effects to be reincorporated without losing analytical control. [J Donier, J Bonart, IM, J-P Bouchaud, arXiv 1412.0141 (2014)]
Model
Let's define the dynamics of the system with some care…
Ingredients

We introduce a latent order book model with the following dynamics:

A. Drift-diffusion: Orders jump left and right at fixed rate. Idiosyncratic component is $D$ and common component is $V_t$ (stochastic zero-mean rate).

B. Cancellations: Orders on the book might be canceled (rate $\nu$)

C. Depositions: Orders are deposed at rate $\lambda$ per unit time on both sides of the book (bids left of price and asks to right)

D. Reactions: Two opposite orders sitting at the same location $x$ can annihilate with rate $\kappa$. We consider $\kappa \rightarrow \infty$.

Everything is constant in space, due to reason which will be clear later.
Remark (I): The difficult term to deal with is $R_{AB}(x,t)$, which is non-linear and affects the average price $\rho_A(p,t)$ or $\rho_B(p,t)$. This definition of price is only meaningful for $\kappa \to \infty$.

The dynamics + a continuous space approximation imply:

$$
\begin{align*}
\frac{\partial A(x,t)}{\partial t} &= -V_A \frac{\partial A(x,t)}{\partial x} + D \frac{\partial^2 A(x,t)}{\partial x^2} - u A(x,t) + \lambda \Theta(x-p_t) - \kappa R_{AB}(x,t), \\
\frac{\partial B(x,t)}{\partial t} &= -V_B \frac{\partial B(x,t)}{\partial x} + D \frac{\partial^2 B(x,t)}{\partial x^2} - u B(x,t) + \lambda \Theta(p_t-x) - \kappa R_{AB}(x,t).
\end{align*}
$$

The previous ingredients + a continuous space approximation imply:

The dynamics
Market clearing and $\varphi$

The reaction term conserves the difference $B-A$, as a consequence of market clearing. We exploit it by defining:

$$\varphi(x, t) = \rho_B(x, t) - \rho_A(x, t)$$

so that

$$\frac{\partial \varphi(x, t)}{\partial t} = -V_t \frac{\partial \varphi(x, t)}{\partial x} + D \frac{\partial^2 \varphi(x, t)}{\partial x^2} - \nu \varphi(x, t) + \lambda \tanh(\mu(p_t - x)),$$

and price is defined by the condition

$$\varphi(x, t) = 0$$

sigmoid deposition

$$\Theta(u) = \frac{1}{1 + e^{-2\mu u}}.$$
A snapshot of a LOB

\[ b(x), \, a(x) \]
Instantaneous shape of the book

\[ \rho_A(x), \, \rho_B(x) \]
Average shape of the book

\[ \varphi(x) \]
Average difference bid-ask
Mechanical reference frame

There is a mean to further simplify the equation. One can reabsorb the drift term with a suitable choice of coordinates:

\[ \hat{p}_t = \int_0^t ds V_s \quad y = x - \hat{p}_t \]

so that

\[ \frac{\partial \varphi(y,t)}{\partial t} = D \frac{\partial^2 \varphi(y,t)}{\partial y^2} - \nu \varphi(y,t) + \lambda \tanh(\mu(p_t - \hat{p}_t - y)) .\]

Remark: Informally, one can interpret this change of coordinates by saying that “information” is factored out…
Stationary behavior

The stationary solution is readily found:

\[ \varphi_{\text{st.}}(y \leq 0) = \frac{\lambda}{\nu} [1 - e^{\gamma y}] ; \quad \varphi_{\text{st.}}(y \geq 0) = -\varphi_{\text{st.}}(-y), \]

for \( \gamma^2 = \nu/D \)

Stationary \( \varphi_{\text{st}}(x) \)

Local linear region!
Zooming-in procedure

Let’s consider the regime in which \( \lambda, \nu \to 0 \) \[ \mathcal{L} = J/D = \lambda \sqrt{D/\nu} \]

\( J \) is the effective current injected at boundaries

Then you get

\[
\frac{\partial \varphi(y, t)}{\partial t} = D \frac{\partial^2 \varphi(y, t)}{\partial y^2}
\]

**Remark (I):** A precise framework in which the *linear approx becomes exact* \( (\lambda \text{ large vol, } \nu \text{ slow dyn}) \)

**Remark (II):** \( J \) fixes number of reactions/transactions per unit time!
Results
Meta-orders

We define a **meta-order** in this framework by adding a term:

\[
\frac{\partial \varphi(y, t)}{\partial t} = D \frac{\partial^2 \varphi(y, t)}{\partial y^2} + m_t \delta(y - y_t)
\]

Representing an extra-flux of orders falling *exactly* on the mid-price.

We also assume:

\[
Q = \int ds \, m_s \quad \text{total volume executed}
\]

\[
m_t = 0 \quad \text{for } t < 0 \quad \text{stationary book at } t=0
\]
Numerical results

We find...

Fig. Evolution in presence of a meta-order $m_t$
Numerical results

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(fig. by J. Donier)

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Fig. Evolution in presence of a meta-order $m_t$
Numerical results

We find…

\[ \lim_{t \to \infty} \rho_t \]

\[ \rho_B \]

\[ \rho_A \]

[fig. by J. Donier]

**Fig.** Evolution in presence of a meta-order \( m_t \)
Numerical results

We find...

Fig. Evolution in presence of a meta-order $m_t$
Modified propagator model

This behavior can be controlled analytically. The book satisfies the equation:

\[
\varphi(y, t) = -\mathcal{L}y + \int_0^t \frac{ds \, m_s}{\sqrt{4\pi D(t-s)}} e^{-\frac{(y-y_s)^2}{4D(t-s)}},
\]

*init. cond*  
*perturbation*

This yields an **exact integral equation** for the price trajectory

\[
y_t = \frac{1}{\mathcal{L}} \int_0^t \frac{ds \, m_s}{\sqrt{4\pi D(t-s)}} e^{-\frac{(y_t-y_s)^2}{4D(t-s)}}.
\]

**Remark:** This generalizes the propagator model (recovered for small perturbations)!
Absence of price manipulation

This modified propagator satisfies nice properties concerning price manipulation. Consider a trajectory $m_t$ and consider

\[
\text{Cost of a closed loop:} \quad \begin{cases} 
C & = \int_0^T ds \, m_s y_s \\
0 & = \int_0^T ds \, m_s 
\end{cases}
\]

Then you have

\[
C = \frac{1}{2} \int_0^T \int_0^T ds ds' m_s \, M(s, s') \, m_{s'} \geq 0
\]

because $M(s, s') = K \, K^\dagger$ is a non-negative operator.
Exact solution for the impact

We have a closed-form solution for constant $m_t = m_0$:

$$y_t = A \sqrt{D t}$$

where the coefficient $A$ satisfies the integral equation

$$A = \frac{m_0}{J} \int_0^1 \frac{du}{\sqrt{4\pi(1-u)}} e^{-\frac{A^2 (1-\sqrt{u})}{4(1+\sqrt{u})}}$$

- Impact is exactly square root
- We have found a non-trivial prefactor
Remarks on the exact solution

- The dependence in $t^{1/2}$ is **generic** (scale invariance!): [see IM, B Toth, J-P Bouchaud (2014)]

- Impact for large executed volumes is *independent of the trading rate*.

- Impact as a function of volume is $\sim Q^{1/2}$

- You recover the Y ratio (for $D \sim \sigma^2/T_{day}$ and $J \sim V/T_{day}$)

\[ \mathcal{I}(Q) = \left( A \sqrt{\frac{J}{m_0}} \right) \left( \frac{Q}{L} \right)^{1/2} \]
Generic trading rates

The property of independence with respect to the trajectory is much more general than that! Consider $m_t$ large. Then

\[ \mathcal{L} y_t | \dot{y}_t | \approx m_t \left[ 1 + D \left( 3 \frac{\ddot{y}_t}{\dot{y}_t^3} - 2 \frac{\dot{m}_t}{m_t \dot{y}_t^2} \right) + O \left( \frac{J^2}{m^2} \right) \right]. \]

(perturbative expansion in powers of $J/m$)

which at first order gives (for $m_t$ positive)

\[ y_t \approx \sqrt{\frac{2}{\mathcal{L}}} \int_0^t ds \, m_s, \]

generalizing previous formula for the impact.
Impact decay

What happens after the trade?

- **Right after**: Steep decay of impact

\[
\frac{\mathcal{I}(Q, T + \epsilon)}{\mathcal{I}(Q)} \to 1 - \sqrt{\frac{\epsilon}{T}}
\]

- **Long after**: Slow relaxation

\[
\frac{\mathcal{I}(Q, t)}{\mathcal{I}(Q)} \to \frac{1}{4} \sqrt{\frac{1}{2\pi}} \frac{m_0 T}{Jt}
\]

We control analytically both limits!
Permanent impact

I didn’t speak about permanent impact yet. **How to account for informed trades?**

\[ \langle V_s m_{s'} \rangle \propto C(s, s') \]

*correlation among drift and executed volume*

This models trades containing/producing information (e.g., alpha).

**Example:** Let’s assume \( C(s,s') = \Gamma e^{-\zeta (s-s')} \). Then we get that

\[
\mathcal{I}_{\text{tot}}(Q, t > T) = \mathcal{I}(Q, t > T) - \frac{m_0 \Gamma}{\zeta} (1 - e^{-\zeta T}) e^{-\zeta (t-T)} + \Gamma Q
\]

*Permanent impact!*
Permanent vs transient impact

Relative contributions of permanent and transient impact

\[ I(Q, t) \]

- \( I_{\text{tot}}(Q, t) \)
- \( I(Q, t) \)
- \( I_{\text{perm}}(Q, t) \)

Permanent impact dominates at large times!

Slow growth at small times!
Discussion
Open problems

❖ The role of fluctuations: the mechanical fluctuations are strongly mean-reverting, and they cross-over at larger times with diffusion due to drift term. Analytically, this is a hard problem.

❖ Fixing the random drift and its correlations with volume: ideally, one would like $m_s$ to be a distribution, endogenized together with the one of $V_s$.

❖ General study of the modified propagator model.
Extensions

- Studying the effect of a distribution on $m_s$
- Fat-tailed statistics for $V_s$
- **Liquidity drought** phenomena: what if orders in the latent order book do not materialize instantly in the real order book?
Conclusions

❖ Market impact follows a universal square root law which seems to depend on few, generic ingredients.

❖ We implement a latent order book model incorporating a one-dimensional notion of price and the market clearing condition.

❖ In this setting, we are able to decompose price in a mechanical and an informational component.

❖ The mechanical part of the impact dominates at small times. We exactly calculate it through an exact integral equation generalizing the propagator model, obtaining (robustly) square root impact.

❖ We reproduce many empirical features of impact: trajectory independence, absence of price manipulation, slow relaxation.
References

