Market Impact and Optimal Execution of Large Trades

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The market impact of trades, i.e. the change in price conditioned on signed trade size, is a key property characterizing market liquidity and is important for understanding price dynamics.

The optimal strategy for an investor with private information about the future price of an asset is to trade incrementally through time (Kyle 1985).

We will call the full orders *metaorders* and the individual trades used to complete the execution *child orders*.

Two (interrelated) research questions:
- Given a model of market impact, what is the optimal execution scheme?
- What are the phenomenological laws describing market impact and price dynamics as a function of volume, volatility, liquidity, etc.?

New results for old problems.
Optimal execution with nonlinear transient market impact

with Gianbiagio Curato and Jim Gatheral
The optimal execution problem depends on the underlying model for price dynamics and on the objective function.

We consider a model of nonlinear transient market impact for the price $S(t)$ of an order execution starting at time $t = 0$ when the price is $S(0) = S_0$ is

$$S(t) = S_0 + \int_0^t f(\dot{x}(s)) G(t - s) \, ds + \int_0^t \sigma \, dW(s),$$

where $\dot{x}(s)$ is the rate of trading, i.e. number of shares per unit of time, at time $s < t$, $f(\dot{x}(s))$ represents the impact of trading at time $s$, and $G(t - s)$ describes the impact decay. Finally, $\sigma$ is the volatility and $W(t)$ is a Wiener process.

This is the continuous time version of the propagator model by Bouchaud et al (2004). See Gatheral (2010). Taranto et al. (2014) for an order book foundation of the model.
The problem

- The optimal execution problem consists in finding the trading strategy \( \Pi = \{ x(t) \}_{t \in [0, T]} \) that minimizes the execution cost for a given total amount \( X \) of shares that we want to trade. The expected cost \( C[\Pi] \) associated with a given strategy \( \Pi \) is given by

\[
C[\Pi] = \mathbb{E} \left[ \int_0^T \dot{x}(t) (S(t) - S_0) \, dt \right] = \int_0^T \dot{x}(t) \int_0^t f(\dot{x}(s)) G(t - s) \, ds \, dt.
\]

and the optimization problem is

\[
\arg \min_{\{x(t)\}} C[\Pi] \quad s.t. \quad \int_0^T \dot{x}(t) \, dt = X
\]

- Trading velocity \( v(t) \equiv \dot{x}(t) \)
- Consistently with empirical data (Lillo et al 2003, Bouchaud et al 2004), we choose

\[
f(v) = \text{sign}(v)|v|^\delta \quad G(t) = t^{-\gamma}
\]

Empirically \( \delta \approx 0.5 \) and \( \gamma \approx 0.5\)
A round trip is an order execution strategy $X$ with $X_0 = X_T = 0$. A price manipulation strategy is a round trip with positive expected revenues.

(Alfonsi et al 2012). A market impact model admits transaction-triggered price manipulation if the expected revenues of a sell (buy) program can be increased by intermediate buy (sell) trades.

A market impact model has negative expected liquidation costs if $E[C] < 0$. 
Proposition

(Klöck et al 2011)

- Any market impact model that does not admit negative expected liquidation costs does also not admit price manipulation.

- Suppose that asset prices are decreased by sell orders and increased by buy orders. Then the absence of transaction-triggered price manipulation implies that the model does not admit negative expected liquidation costs. In particular, the absence of transaction-triggered price manipulation implies the absence of price manipulation in the usual sense.

Proposition

(Gatheral 2010). If \( G(t) = t^{-\gamma} \) and \( f(v) = \text{sign}(v)|v|^\delta \), absence of price manipulation imposes that

\[
\gamma + \delta \geq 1 \quad \quad \quad \quad \gamma \geq \gamma^* = 2 - \frac{\log 3}{\log 2} \approx 0.415
\]
Transforming the cost minimization into an integral equation

- Dang (2014) shows that, given $f \in C^1(\mathbb{R})$ and $G \in L^1[0, T]$, for the class of functions $x$ on $[0, T]$ satisfying
  - $x$ is absolutely continuous on $(0, T)$,
  - $f \circ x \in L^1[0, T],$
the following necessary condition for the stationarity of the functional of eq. 3 holds:

$$\int_0^t f(v(s)) G(t-s) \, ds + f'(v(t)) \int_t^T v(s) G(s-t) \, ds = \lambda, \quad (7)$$

where $\lambda$ is a constant set by the constraint equation.

- In the concave ($\delta < 1$) case there is no guarantee that the minimum is global

- This is a weakly singular Urysohn integral equations of the first kind (very hard to solve!!)
Equation 7 is a weakly singular Urysohn equations of the first kind

\[ \int_{0}^{T} G(|t - s|) F(v(s),\ t) \, ds = \lambda \]  

where

\[ F(v(s),\ t) = \begin{cases} f(v(s)), & s \leq t \\ v(s) f'(v(t)), & s > t, \end{cases} \]

- Two nonlinearities: one related to \( f \) and one related to \( F \).
- The term with the first derivative entangles the susceptibility of response at time \( t \) with the future trading rates, i.e. a coupling between present and future values of \( v \). This implies that Eq. 9 cannot be classified as a weakly singular nonlinear Fredholm equation, because the function \( F \) depends both on \( t \) and on \( s \).
- For concave \( f \), the integral equation becomes meaningless when \( v = 0 \).
In the linear case, $f(\nu) = \nu$, the integral equation becomes a Fredholm integral equation of the Abel type, which can be solved analytically (Gatheral et al. 2012, see Busseti and Lillo (2012) for an empirical calibration) as

$$\nu(t) = \frac{c}{[t(T - t)]^{1/2}},$$  \hspace{1cm} (10)

where $c$ is uniquely determined by the constraint Eq. 4 as

$$c = X/\left(\sqrt{\pi} \left(\frac{T}{2}\right)^{\gamma} \frac{\Gamma((1 + \gamma)/2)}{\Gamma(1 + \gamma/2)}\right),$$  \hspace{1cm} (11)

where $\Gamma(\cdot)$ is the Euler’s function.

This solution has a U shape and is symmetric under time reversal, i.e. $\nu(t) = \nu(T - t)$, $t \in [0, T/2]$.

In the following we will refer to this solution as the GSS solution.
The fixed point method by Dang

Recently Dang (2014) proposed a fixed point iteration scheme to find a numerical solution by quadrature methods.

Figure: Left panel: convergence region of the Dang’s fixed point method on the parameter space \((N, \delta - 1)\). Right panel: squared residual error of solutions obtained as map’s fixed points.

We find that the the fixed point method seems to converge only for moderate non linearity and/or moderate discretizations of the \([0, T]\) interval.
Perturbative approach

Figure: Solution of the weak nonlinear Fredholm integral equation for $\gamma = 0.5$, $\epsilon = 0.02$ and $X = 0.1$. The full line represents the solution $v(s) = u(s) + \epsilon w(s)$. The solution is not symmetric for time reversal. The dotted line represents the GSS solution, i.e. the solution valid for the linear impact case.

We perform an expansion $f(v) = v^{1-\epsilon}$, with $0 < \epsilon \ll 1$ and we solve exactly the perturbed equation.

No symmetry for time reversal: front loading for concave impact ($\delta < 1$), back loading for convex impact ($\delta > 1$).
Homotopy Approach

- We solve the integral equation by means of the Discrete Homotopy Analysis Method (DHAM), i.e. as a continuous transformation of a given solution (never crossing $v = 0$).
- Given the following general equation
  \[ N[v(t)] = 0, \quad (12) \]
  we construct the so-called zero-order deformation equation
  \[ (1 - p) L \left[ \phi(t; p) - v^0(t) \right] = p \bar{h} H(t) N[\phi(t; p)], \quad (13) \]
  where $p \in [0, 1]$ and $v^0(t)$ is an initial guess
- \[ \phi(t; 0) = v^0(t), \quad \phi(t; 1) = v(t). \quad (14) \]
- Expanding $\phi(t; p)$ in Maclaurin series with respect to $p$, we have
  \[ \phi(t; p) = v^0(t) + \sum_{m=1}^{\infty} v^m(t) p^m, \quad (15) \]
  where
  \[ v^m(t) = \frac{1}{m!} \frac{\partial \phi^m(t; p)}{\partial p^m} \bigg|_{p=0}. \quad (16) \]
Homotopy Approach for market impact

• The operator is

\[ \mathcal{N}[v(t)] = -\lambda + \int_0^T G(|t-s|) F(v(s), t) \, ds. \]  \hspace{1cm} (17)

• Choosing \( \mathcal{L} \) and \( H(t) \) as the identity operators, the zero-order deformation equation is

\[ (1 - p) \left[ \phi(t; p) - v^0(t) \right] = \hbar v \left( \mathcal{N}[\phi(t; p)] \right). \] \hspace{1cm} (18)

• Differentiating \( m \) times we get

\[ v^m(t) = v^{m-1}(t) + \hbar R^m \left( v^{m-1} \right), \] \hspace{1cm} (19)

where for \( m > 1 \)

\[ R^m \left( v^{m-1} \right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(t; p)]}{\partial p^{m-1}} \bigg|_{p=0} \]

\[ = \int_0^T G(|t-s|) \left\{ \frac{1}{(m-1)!} \frac{\partial^{m-1} F(\phi(s;p), t)}{\partial p^{m-1}} \bigg|_{p=0} \right\} \, ds. \] \hspace{1cm} (20)
Figure: The logarithm of the squared residual $\mathcal{E}^7(\bar{h})$ is illustrated on the left panel, the minimum is attained for $\bar{h} = -55.7$ where we have $\mathcal{E}^7 = 3.2 \times 10^{-6}$. The GSS guess and the DHAM solution are reported on the right panel respectively by a full green line with circles and a dashed blue line with circles, are reported also the results of the seven deformation equations.
### Discrete Homotopy Approach: Costs

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**Table:** Costs for three different strategies, VWAP, GSS, and DHAM, in the no-dynamic-arbitrage region for $\gamma = 0.45, 0.5$. The numbers in boldface are those achieving the smallest cost. The difference between costs increases with the degree of non-linearity, i.e. $\delta < 1$. In this case we use a GSS initial guess to obtain the DHAM solution.

The DHAM solution has a cost up to 20% smaller than the GSS, while the latter has a cost which is only 1% smaller than the VWAP.
Fully numerical solutions

- DHAM solution is a continuous deformation of a VWAP or a GSS solution
- Therefore it is smooth and, more important, has always the same sign
- What happens if we minimize **numerically** the cost on a discrete grid of $N$ intervals in $[0, T]$ (piecewise constant solution)?

\[
\arg\min_{v} \sum_{i=1}^{N} \sum_{j=1}^{N} v^n_i \cdot f(v^n_j) \cdot A_{ij} \quad \text{s.t.} \quad \sum_{i=1}^{N} v_i = \frac{NX}{T} \quad (21)
\]

where the $A_{ij}$ are elements of a Toeplitz matrix that describes the decay kernel $G(t - s)$

\[
A_{ij} = 0; \ j > i, \\
A_{ii} = \frac{1}{(1 - \gamma)(2 - \gamma)} \left( \frac{T}{N} \right)^{2-\gamma}; \\
A_{ij} = \frac{1}{(1 - \gamma)(2 - \gamma)} \left( \frac{T}{N} \right)^{2-\gamma} \left\{ (i - j + 1)^{2-\gamma} - 2(i - j)^{2-\gamma} + (i - j - 1)^{2-\gamma} \right\};
\]
A $N = 2$ period motivating example

Let us consider the case of a buy program over $N = 2$ periods

![Cost function graph](image)

**Figure:** Cost function $C[v_1, 2X - v_1]$ for $X = 0.1, \gamma = 0.5$. For $\delta = 1$ the minimum is at $v_1 = X$. In the nonlinear case there are two local minima.

When nonlinearity is strong, it is optimal to sell in the first period -> transaction triggered manipulation!!
SQP numerical optimization

- We perform an in-depth numerical optimization of the cost function when $N$ is large ($\sim 100$)
- We use Sequential Quadratic Programming (SQP) with a large number of starting points on the simplex $\sum_{i=1}^{N} v_i = NXT^{-1}$
- We find a very large number of distinct extremal points and we select the one with the smallest cost
- We use second order condition to verify that a very large fraction of extremal points are minima
- By computing the eigenvalue spectra of the Hessian of the cost function at the minima, we exclude that the landscape of cost is sloppy (i.e. does not depend strongly on a few number of directions in the state-space).
- The landscape is in fact rugged (i.e. composed by many local minima with similar cost)
- Many suboptimal minima correspond to similar trading patterns (see below)
The "optimal" solution for a buy program

Figure: Optimal solution given by the SQP-algorithm for a buy-program where $X = 0.1$, i.e. 10% of a unitary market volume. We report the volume to be traded in each interval of time.

Under strong non-linearity, the optimal buy program is composed by few intense buying periods interspersed by long weak selling periods -> transaction triggered manipulation
Figure: Costs for a buy-program where $X = 0.1$, when $N = 150$. We compute the costs on the no-dynamic-arbitrage zone. Holding $\gamma$ fixed, the cost relative to the global minima is not a monotonic function of $\delta$. Negative costs can be present for $\gamma < 0.5$.

Negative cost for strong non-linearity !! (and large discretization)
Price manipulation and its regularization

- The no-arbitrage condition $\delta + \gamma > 1$ is not sufficient to guarantee the absence of price manipulation.
- Any non-linear impact leads to price manipulation strategies for sufficiently large discretization.
- It is possible to regularize the solutions with two approaches:
  - Adding a spread cost
    \[
    C = \sum_{i=1}^{N} \sum_{j=1}^{N} v_i f(v_j) A_{ij} + \delta_S \frac{T}{N} \sum_{i=1}^{N} |v_i|, \tag{23}
    \]
    where $\delta_S$ is half the bid-ask spread. This is equivalent to a $L_1$ or LASSO regularization widely used in computer science.
  - Modifying the impact function $f(v)$ to
    \[
    f_G(v) = c \text{ sign}(v) \left\{ \left( \frac{|v|}{|v| + V} \right)^\delta + d \frac{|v|(|v| + V)}{V^2} \right\}, \tag{24}
    \]
    where $V = \frac{X_M}{T}$ is the market volume per unit time. This is concave for small $v$ and convex for large $v$ (illiquidity wall)
- Both regularization succeed (in some parameter regime) to avoid negative costs
The larger the spread, the stronger the regularization, the smaller the contribution from sell trades.

By imposing the constraint $v_i \geq 0$, we obtain that the optimal solution is trading in bursts.
Figure: Top. Concave-convex impact function for values of parameters: $c = 1$, $\delta = 0.55$, $X_M = 1$, $T = 1$. Bottom. Optimal solution given by the SQP-algorithm for a buy-program
Beyond the square root: Evidence for logarithmic dependence of market impact on size and participation rate

with Elia Zarinelli, Michele Treccani, Doyne Farmer

arXiv:1412.2152
Market impact laws

- Kyle’s original model (1985) predicts that price impact should be a linear function of the metaorder size.
- Empirical studies have consistently shown that the price impact of a metaorder is a non-linear concave function of its size.
- Market impact $I$, i.e. the expected average price change between the beginning and the end of a metaorder of size $Q$ is empirically fit by

$$I(Q) = \pm Y \sigma_D \left( \frac{Q}{V_D} \right)^\delta$$

where $\sigma_D$ is the daily volatility of the asset, $V_D$ is the daily traded volume, and the sign of the metaorder is positive (negative) for buy (sell) trades. The numerical constant $Y$ is of order unity and the exponent $\delta$ is in the range 0.4 to 0.7, but typically very close to $1/2$, i.e. to a square root.
- This is the **square-root impact law** (Barra 1997, Almgren et al 2005, Moro et al 2009, Toth et al 2011, Bershova et al 2013)
A portfolio manager liquidates a position and splits its order between brokers.

A broker receives orders from different portfolio managers and bundles them in a unique metaorder.

Investment Fund
- Portfolio manager 1
- Portfolio manager 2
- Portfolio manager 3

1_V
- 1_V_1 → Broker 1
- 1_V_2 → Broker 2
- 1_V_3 → Broker 3

Investment Fund
- Portfolio manager 1
- Portfolio manager 2
- Portfolio manager 3

1_V_1
- 2_V_1 → Broker 1
- 3_V_1 → Broker 1

Market
- V_1
**Ancerno Dataset - 2**

- **Metaorder** definition: an execution performed by a single Broker, on a single stock, in a given direction. All metaorders are completed within a trading day.

- The dataset is heterogeneous, containing metaorders traded by many financial institutions for different purposes and it spans several years.

- For each metaorder in the dataset we recover the relative **daily fraction** $\pi$, the **participation rate** $\eta$, and the **duration** $F$. By exploiting the price data, we also recover the time series of the price during and after the execution of the metaorder.

- We introduce the following filters:

| Filter 0 | Selecting metaorders traded between January 2007 and December 2009 | \( \sim 28,500,000 \) |
| Filter 1 | Selecting metaorders traded on Russell3000 index | \( \sim 23,000,000 \) |
| Filter 2 | Selecting metaorders traded during regular trading section: 09:30 - 16:00 | \( \sim 11,000,000 \) |
| Filter 3 | Selecting metaorders with duration longer than 2 minutes | \( \sim 7,500,000 \) |
| Filter 4 | Selecting metaorders whose participation rate is smaller than 0.3 | \( \sim 7,000,000 \) |

\[
\begin{align*}
\text{Sign} & : \quad \epsilon = \pm 1 \\
\text{Duration} & : \quad F := \frac{V_P}{V_D} \\
\text{Participation rate} & : \quad \eta := \frac{Q}{V_P} \\
\text{Daily rate} & : \quad \pi := \frac{Q}{V_D} \\
\text{Trading profile} & : \quad \rho(v, v_s, v_e) \\
\text{Not available information} & : \quad \text{(Redacted)}
\end{align*}
\]
A snapshot of the market

Time series of metaorders active on the market for AAPL in the period March-April 2008. Buy (Sell) metaorders are depicted in blue (red). The thickness of the line is proportional to the metaorder participation rate. More metaorders in the same instant of time give rise to darker colours. Each horizontal line is a trading day. We observe very few blanks, meaning that there is almost always an active metaorder from our database, which is of course only a subset of the number of orders that are active in the market.
The participation rate $\eta$ and the duration $F$ are both well approximated by a truncated power-law distribution over several orders of magnitude.

**Duration**

$F := \frac{V_P}{V_D}$

**Participation rate**

$\eta := \frac{Q}{V_P}$

$$f(x) = C \cdot x^a$$

$C = 0.220$ $a = -0.932$

$C = 0.223$ $a = -0.864$
The price impact curve: excess concavity

Price impact curve: the average relative price change between the end and the beginning of the execution, conditioning on the daily rate $\pi := Q/V_D$

$$\mathcal{I}(\pi) := \mathbb{E} \left[ \epsilon (s(v_e) - s(v_s)) \middle| \pi \right]$$

$s(v) := \log S(v) / \sigma_D$  Rescaled price

A square-root model well describes price impact only in the central region (red curve).

A logarithmic (more concave) model allows to capture the whole shape of the curve (blue curve).
Further conditioning…

**Trading year**

**Stock market capitalisation**

The price impact curve is **quite stable** and, with the exception of the small capitalisation conditioning, the logarithmic function **always better** explains the data.
The price impact surface

Price impact surface: the average relative price change between the end and the beginning of the execution, conditioning on the participation rate \( \eta := Q/V_P \) and the duration \( F := V_P/V_D \)

\[ \mathcal{I}(\eta, F) := \mathbb{E} \left[ \epsilon (s(v_e) - s(v_s)) \middle| \eta, F \right] \]

A double logarithmic function (blue surface) better describes the data compared with a double power-law functions (yellow surface)

It is possible to recover the price impact curve by averaging on regions such that \( \pi = \eta F \) is constant (diagonal lines in the double logarithmic scale)
Analysis of the residuals

Price impact models:

\[ f(\eta, F|Y, \delta, \gamma_1) = Y \cdot \eta^\delta \cdot F^{\gamma_1} \]
\[ g(\eta, F|a, b, c) = a \cdot \log_{10}(1 + b\eta) \cdot \log_{10}(1 + cF) \]

Residuals of the fitted models:

Double power-law: a clear non-random pattern emerges: positive residuals in the centre, negative residuals in the periphery

Double logarithm: residuals are evenly distributed
The price impact trajectory - during the execution

We fix the participation rate and the duration of metaorders, and we follow the price impact trajectory during the execution:

\[
\mathcal{I}(v|\eta, F) := \mathbb{E} \left[ \epsilon \left( \tilde{S}(v) - \tilde{S}(v_s) \right) \right] | \eta, F
\]

The price impact trajectories (lines) deviate from the price impact curve/surface (circles)

The price impact trajectories revert during the execution of the metaorder
Inferring average execution pattern

Price dynamics - Gatheral model

Continuous-time stock price model for a trader who can move the price of the asset. As long as the trader is not active, the asset price is determined by the other market participants and follows a brownian motion.

\[
\tilde{S}(v_e) = \tilde{S}(v_s) + \epsilon \int_{v_s}^{v_e} f(q(s))G(v_e - s)ds + \int_{v_s}^{v_e} dW_s
\]

Price impact of a metaorder - Gatheral model

\[ f(q) \sim q^\delta \quad \text{Single-trade impact function} \]
\[ G(s) \sim s^{-\gamma} \quad \text{Kernel memory} \]
\[ \alpha \quad \text{Execution risk aversion} \]
Permanent impact (using all metaorders)

Figure: Decay of temporary market impact after the execution of the meteorder. We follow the normalised market impact path $I_{ren}(z)$ as a function of the rescaled variable $z = v/F$, without conditioning on any variable. The market impact path of each metaorder is followed also in the following day. The red horizontal line corresponds to 2/3, as predicted by the model of Farmer et al. 2013.

On average the 2/3 prediction of the model of Farmer, Gerig, Lillo, and Waelbroeck (2013) is satisfied.
The price impact trajectory - after the execution

Decay of temporary price impact after the execution of the meteorder. We follow the normalised price impact as a function of the rescaled variable \( z = v/F \).

For small participation rates, the price impact trajectories of longer metaorders relax to levels which are higher than those of shorter metaorders.

For large participation rates, the price impact trajectories superimpose quite well one with each other. They relax very slowly and we do not observe any flattening of the curve. Quite interestingly, in this regime the price impact trajectories are well described by the prediction of the propagator model (black curve).
The role of metaorder sign autocorrelation

The picture emerging from the previous analysis can be partly clarified by taking into account the **autocorrelation of the sign of metaorders**. If different metaorders executed consecutively or in the same time period typically have the same sign, we expect that the effect of this correlation is to keep the price impact of a single metaorder relaxation **artificially high**. Moreover the effect of autocorrelation is

- **stronger** for longer metaorders, since the probability of overlapping with other metaorders is larger, and for lower participation rates, since their effect on price can be considerably perturbed by metaorders with larger participation rates.

- **milder** for shorter metaorders, because of the lower probability of overlap, and larger participation rates, because the effect of metaorders with lower participation rates on price becomes negligible.

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<th>a.n.o.m.</th>
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Average fraction of overlapping metaorders with the same sign
Conclusions

- Optimal execution under nonlinear transient impact is a hard problem with a rugged landscape.
- Constraining to buy for buy metaorders, the DHAM optimal execution outperforms significantly other executions.
- However, nonlinear transient impact has executions with negative cost, i.e., possibility of price manipulation.
- Regularization can help.
- Logarithmic market impact as a function of volume.
- (Logarithmic) Market impact surface rather than market impact curve.
- Transient impact is not tracking temporary impact: due to risk-averse executions.
- Average permanent impact is $2/3$ of the temporary impact but the level depends on metaorder duration and participation rate.
- Role of herding among metaorders in the permanent impact.